Dynamical systems approach to Saffman-Taylor fingering: Dynamical solvability scenario

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A dynamical systems approach to competition of Saffman-Taylor fingers in a Hele-Shaw channel is developed. This is based on global analysis of the phase space flow of the low-dimensional ordinary-differential-equation sets associated with the classes of exact solutions of the problem without surface tension. Some simple examples are studied in detail. A general proof of the existence of finite-time singularities for broad classes of solutions is given. Solutions leading to finite-time interface pinchoff are also identified. The existence of a continuum of multifinger fixed points and its dynamical implications are discussed. We conclude that exact zero-surface tension solutions taken in a global sense as families of trajectories in phase space are unphysical because the multifinger fixed points are nonhyperbolic, and an unfolding does not exist within the same class of solutions. Hyperbolicity (saddle-point structure) of the multifinger fixed points is argued to be essential to the physically correct qualitative description of finger competition. The restoring of hyperbolicity by surface tension is proposed as the key point to formulate a generic dynamical solvability scenario for interfacial pattern selection.

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I. INTRODUCTION

The Saffman-Taylor (ST) problem [1–4] has played a central role for several decades as a prototype system in the study of interfacial pattern formation [5–9], particularly concerning the issue of pattern selection [1,10–12]. Despite its elongated existence, the problem continues to pose new challenges with the focus now on its *dynamical* aspects. In this sense, the ST problem is becoming instrumental once more in gaining insights into the possibly generic behavior, due to its relative simplicity in the context of morphologically unstable interfaces in nonequilibrium systems.

A full understanding of the analytical mechanisms leading to steady state selection by surface tension as a singular perturbation in the problem was not completely achieved until the late 1980s [13–17] and the resulting scenario, usually referred to as microscopic solvability (MS) [5,6], has currently become a paradigm for many other systems, for instance, in free dendritic growth [7,8]. Such solvability analysis, however, is strictly *static*, in the sense that it is concerned with the existence and linear stability of stationary solutions. The importance of dynamics in the process of selection was pointed out in Refs. [18–20] where it was argued that the Saffman-Taylor finger solution was not the universal attractor of the problem if the displacing fluid has a non-negligible viscosity. More recently, the traditional MS scenario of selection has not been free from some controversy in connection with the dynamics of the zero surface tension problem [21–26]. The singular effects of surface tension on the dynamics have been pointed out as a rather subtle and challenging issue [27–29] and the possibility of some extension of the MS scenario of selection to the *dynamics* has been suggested [4,25,30,31]. In any case, the study of the dynamics of morphologically unstable interfaces in the context of Laplacian growth or, more generally, of diffusion-limited growth of interfaces in nonequilibrium conditions, has been rather elusive to analytical treatment due to the highly nonlinear and nonlocal character of the equations. For the viscous fingering problem, the extent to which the case of zero surface tension does capture the physics of the fingering dynamics remains a poorly understood yet fundamental issue, particularly given the availability of exact solutions in that limiting case.

The present paper expands and elaborates in depth the approach first introduced in Ref. [30], which is based on the ideas and concepts of dynamical systems (DS) theory. With this general point of view, we study in detail some specific classes of solutions of the zero surface tension problem, with focus on the qualitative (topological) properties. As we will see, the comparison of the problem with and without surface tension is essentially qualitative in nature, so it is important to pose questions in a framework that is at the same time qualitative and mathematically precise. Such framework is the theory of dynamical systems. The use of this conceptual tool will help us formulate precise questions to which we can give an answer. From the above results and within this spirit, we will reformulate the issue of a possible extension to *dynamics* of the MS scenario of steady state selection, and suggest a possible answer to that.

The common understanding of the finger competition process (sometimes referred to as finger coalescence) leading to the selected steady state is usually based on qualitative screening arguments. In some cases these have been shown to be too naive [30], particularly in the light of the recent findings of stationary solutions with nonzero surface tension but with coexisting unequal fingers [31]. To gain insights into the dynamics of finger competition it seems natural to turn to the idealized (zero surface tension) problem. Despite the fact that the zero-surface tension ST problem is ill posed as an initial-value problem [27], the crucial fact that makes the idealized problem attractive to analytical treatment is the availability of rather broad classes of explicit time-dependent solutions [32–35]. Some classes of solutions are known to develop finite-time singularities in the form of cusps and are thus not of much interest in the physics of viscous fingering, since surface tension regularization will obviously remove such singularities. Nevertheless, a still remarkably large class of known solutions is free from singularities and therefore physically acceptable, *in principle*. The basic question is then what would be the effect of regularization (introducing a small but nonzero surface tension) to those solutions. This question was first raised in Ref. [36] where it was shown that for some classes of initial conditions, the effect of surface tension as a perturbation could be considered as basically regular, while for other initial conditions the singular character of the perturbation showed up dramatically in the dynamics. In other configurations, such as for circular geometry, surface tension has also been shown to behave as a regular perturbation [37]. Indeed, in view of the morphological diversity that is included in the known nonsingular solutions, one may be tempted to expect that, since such solutions remain smooth for all the time evolution, they should stay close to the solutions of the regularized problem as $d_0\to 0$ for a time lapse that would increase with decreasing d_0. Siegel and Tanveer [28] and Siegel *et al.* [29] have shown that this is *not* the case, and, in general, the idealized and the regularized solutions differ significantly from each other at order one time. In the remarkable contribution of Refs. [28,29], however, only simple examples of single-finger evolutions are considered, so the extent to which those conclusions can be extended to multifinger configurations still requires a careful analysis [38]. Furthermore, even though the idealized and the regularized solutions differ significantly after a time of order unity (basically independent of surface tension), one could still argue that the qualitative evolution may be basically unaffected by surface tension if the finger width is not too different from the selected one in the regularized case. Therefore, the possibility that some classes of solutions or some particular dynamic mechanisms are basically insensitive to surface tension remains open.

Following Ref. [30], we will exploit the fact that the integrable classes of initial conditions define finite-dimensional invariant manifolds of the full (infinite-dimensional) problem, so it makes sense to study the resulting low-dimensional dynamical systems and compare them with properly defined finite-dimensional subsets of the regularized problem. With this analysis we will clarify in what precise sense the nonsingular exact solutions of the idealized ST problem are, in general, unphysical. Once settled the unphysical nature of a broad class of solutions, a natural question to address in whether a selection principle is associated with the surface tension regularization, which can be understood as a dynamical generalization of the MS scenario. We will address this point in the light of our results and discuss how and in what sense such dynamical MS can be formulated.

The rest of the paper is organized as follows. In Sec. II the equations describing Hele-Shaw flows in channel geometry are recalled, together with the conformal mapping formulation. The characterization of finger competition is described and the dynamical systems approach to the problem is introduced. In Sec. III the minimal class presented in Ref. [30] is revisited. In Secs. IV and V various generalizations of the minimal class are introduced. In Sec. VI we discuss the precise role of zero surface tension solutions and their relevance to an understanding of the dynamics of Hele-Shaw flows. A dynamical solvability scenario is proposed and discussed as a generalization of MS theory. Finally, in Sec. VII we summarize our main results and conclusions.

II. FORMULATION OF THE PROBLEM AND DYNAMICAL SYSTEMS APPROACH

Consider a Hele-Shaw cell of width W in the y direction and infinite length in the x direction, with a small gap b between the plates. The fluid flow in this system is effectively two dimensional and the velocity $\mathbf{v}$ obeys Darcy's law,

$$\mathbf{v}=-\frac{b^2}{12\mu}\boldsymbol{\nabla}p, \tag{1}$$

where p is the fluid pressure and μ is the viscosity. We define a velocity potential $\varphi=-(b^2/12\mu)p$, and assuming that the fluid is incompressible ($\boldsymbol{\nabla}\cdot\mathbf{v}=0$) we obtain the bulk equation to be the Laplace equation $\nabla^2\varphi=0$. This must be supplemented with the two boundary conditions $\varphi|_\Gamma=(b^2\sigma/12\mu)\kappa$ and $v_n=\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\varphi$, where Γ means that the quantity is evaluated on the interface, v_n is the normal component of the velocity of the interface, κ is the curvature, $\hat{\mathbf{n}}$ is the unit vector normal to the interface and σ is the surface tension. We define a dimensionless surface tension parameter d_0 as $d_0=\sigma b^2\pi^2/12\mu V_\infty W^2$, where V_∞ is the fluid velocity at infinity. For simplicity we assume periodic boundary conditions at the sidewalls of the channel, and we will see that nothing essential is lost with respect to competition in a rigid-wall channel.

We use conformal mapping techniques to formulate the problem [2]. We define a function $f(\omega,t)$ that conformally maps the interior of the unit circle in the complex plane ω into the viscous fluid in the physical plane $z=x+iy$. We assume an infinite channel in the x direction. The mapping $f(\omega,t)$ must satisfy $\partial_\omega f(\omega,t)\neq 0$ inside the unit circle, $|\omega|\leqslant 1$. Moreover, it has the form

$$f(\omega,t)=-\ln\omega+h(\omega,t), \tag{2}$$

where $h(\omega,t)$ is an analytic function in the whole unit disk. We define the complex potential as the analytic function $\Phi=\varphi+i\psi$, where the harmonic conjugate ψ of φ is the stream function. The width of the channel is $W=2\pi$ and the velocity of the fluid at infinity is $V_\infty=1$. It can be shown that the evolution equation for the mapping $f(\omega,t)$ for zero surface tension reads

$$\mathrm{Re}\{i\partial_\phi f(\phi,t)\partial_t f^*(\phi,t)\}=1. \tag{3}$$

The conformal mapping formulation of the problem with finite surface tension can be found for instance in Ref. [4].

Let us recall some ideas and definitions introduced in Refs. [30,4]. To quantify finger competition it is useful to define individual growth rates of fingers, as the peak-to-peak difference of the stream function between the maximum and the minimum that are adjacent to the finger tip [20]. According to this definition, one is assigning a nonzero growth rate to a finger if it advances faster than the mean interface. Looking at individual growth rates one can easily distinguish two different stages in the process of finger competition. A first stage characterized by the monotonic growth of all finger growth rates and a second one dominated by the redistribution of the total growth rate among the fingers. We call these two stages *growth* and *competition* regimes, respectively. For two-finger configurations, during the growth regime the two fingers develop from small bumps of the initially flat interface, while the total growth rate $\Delta\psi_T(t) = \Delta\psi_1(t) + \Delta\psi_2(t)$ increases until it reaches a value close to its asymptotic value $\Delta\psi_T(\infty)$. The decrease in the growth rate of one of the fingers signals the outcome of the competition regime: there is a redistribution of flux from one finger to the other one. We also define the existence of *successful* competition as the ability to completely suppress the growth rate of one finger.

The theory of dynamical systems is a mathematical discipline for studying ordinary differential equations or flows (and also difference equations or maps) with stress on geometrical and topological properties of families of solutions [39]. Such global approach seems thus appropriate to study in a precise way the qualitative properties of our problem. An important concept in dynamical systems theory is that of *structural stability*, which captures the physically reasonable requirement of robustness of the mathematical description to slight changes in the equations. Roughly speaking, a system is said to be structurally stable if slight perturbations of the equations yield a topologically equivalent phase space flow [39]. When a DS depends on a set of parameters, the bifurcation set is defined as those points in parameter space where it is structurally unstable. In this case the structural instability at an isolated point in parameter space is the property necessary for the system to change its qualitative behavior. At a bifurcation point, adding perturbations to the equations to make the system structurally stable is called an *unfolding* [39]. For dimensions higher than two, the mathematical definition of structural stability is usually too stringent. For the purposes of the present discussion and most physical applications it is sufficient to consider the notion of *hyperbolicity* of fixed points, which in two dimensions is directly associated with structural stability through the Peixoto theorem [39]. A fixed point is hyperbolic when the linearized flow has no marginal directions, that is, all eigenvalues of the linearized dynamics are nonzero. We will see that the nonhyperbolicity of the double-finger fixed point (in general the *n*-equal-finger fixed point) and the nonexistence of an unfolding of it within the known class of solutions is at the heart of the unphysical nature of this class of solutions.

A dynamical systems approach to the Saffman-Taylor problem, however, must deal with an infinite-dimensional problem in an unbounded domain. The usual dimension-reduction techniques such as center manifold projection are of no use in studying the strongly nonlinear dynamics of competing fingers, since generically the system is far from threshold and the growth does not saturate to finite amplitudes. A weakly nonlinear analysis is still possible but limited to a rather early transient [40]. As an alternative, the basic point that we will exploit here is the fact that all exact solutions known explicitly for the idealized problem ($d_0 = 0$) are defined in terms of ordinary differential equations (ODE's) for a finite number of parameters, and thus define finite-dimensional DS's in the phase space defined by those parameters. We will denote the DS defined by the complete ST problem (finite d_0) in an infinite-dimensional phase space as $S^\infty(d_0)$. The limit $d_0 \to 0$ defines a limiting DS that we will refer to as $S^\infty(0^+)$, which, as we will see, does not coincide with $S^\infty(0)$.

The phase space may be parametrized, for instance, using the coefficients of the Taylor expansion of the analytical part $h(\omega)$ of the conformal mapping. The explicit (infinite) set of ODE's for them are obtained inserting the expansion in the evolution equation for the mapping. In the case of strictly zero surface tension, this set may be solved exactly for some classes of initial conditions. These define invariant manifolds of $S^\infty(0)$ of *finite* dimension. In this context, finding explicit solutions implies identifying a specific analytic structure of $h(\omega)$, with a finite number of parameters, which is preserved under the time evolution. If this condition is fulfilled, then a set of ODE's for those parameters can be closed, and defines a certain DS on a finite-dimension space. Of special relevance are the classes of solutions that may remain smooth (nonsingular) for all the time evolution. The most important one for the present purposes takes the general form [32,33]

$$h(\omega) = d(t) + \sum_{j=1}^{N} \gamma_j \ln[1 - \alpha_j(t)\omega], \tag{4}$$

where γ_j are constants of motion with the restriction $\sum_{j=1}^{N}\gamma_j = 2(1-\lambda)$, where λ is the asymptotic filling fraction of the channel occupied by fingers. If all γ_j are real the evolution is free of finite-time singularities, and if any γ_j has an imaginary part then finite-time singularities may appear for some set of initial conditions (see Sec. V C). Inserting this ansatz in Eq. (3) a closed set of ODE's for the finite number of parameters $\alpha_j(t)$ can be found. The region that is physically meaningful is the one in which $|\alpha_j| \le 1$ (including the equal sign allows for the limiting case of infinite fingers, and makes the phase space compact). The DS defined by Eq. (4) in the $2N$-dimensional hypervolume will be denoted as $L^{2N}(\{\gamma_j\})$. Notice that modifying the parameters $\{\gamma_j\}$, which are constants of motion under the dynamics defined through Eq. (3), corresponds to varying initial conditions in the phase space of $S^\infty(0)$, while, from the viewpoint of the finite-dimensional DS's denoted by $L^{2N}(\{\gamma_j\})$, it corresponds to changing the DS itself, that is, changing the ODE's obeyed by the dynamical variables. In this sense, $\{\gamma_j\}$ label a set of DS's defined on a $2N$-hypervolume $|\alpha_j| \le 1$.

III. THE TWO-FINGER MINIMAL MODEL

A. The model

The simplest class of exact time-dependent solutions of Eq. (3) containing the three physically relevant fixed points—the planar interface (PI), the single Saffman-Taylor (1ST) fixed point, and the double Saffman-Taylor (2ST) fixed point—was introduced in Ref. [30] and reads

$$f(\omega,t) = -\ln\omega + d(t) + (1-\lambda)\ln[1-\alpha(t)\omega] + (1-\lambda)\ln[1+\alpha(t)^*\omega], \tag{5}$$

where λ is a real-valued constant in the interval [0,1], $\alpha(t) = \alpha'(t) + i\alpha''(t)$ and $d(t)$ is real. The relevant phase space for a given λ is the first quadrant of the unit circle in the (α', α'') space. The other three quadrants describe interface configurations that are equal or symmetrical to the interfaces contained in the first quadrant. In this section we will summarize the basic results discussed in detail in Refs. [30,4], and put them in the more general perspective of the following sections. The interface described by this mapping consists generically of two unequal fingers, axisymmetric and without overhangs. The case $\alpha'(t)=0$ gives the time-dependent ST finger solution, and $\alpha''(t)=0$ corresponds to the double time-dependent ST finger. For $|\alpha(t)| \ll 1$ the interface consists of a sinusoidal perturbation of the planar interface.

The phase portraits of the dynamical systems defined by the solutions of the form Eq. (5) for different λ were studied in detail in Refs. [30,4]. The most salient feature was that the basin of attraction of the Saffman-Taylor single finger is not the whole phase space. The separatrix between the basin of attraction of the ST finger and the rest of the flow starts in the planar interface fixed point and ends in a new fixed point whose location depends on λ. The flow not attracted to the single-finger fixed point, evolves to a continuum of fixed points, corresponding to stationary solutions with two unequal fingers advancing with the same velocity. The basin of attraction of the ST finger was shown to be larger for smaller λ but never the full phase space. For $\lambda = 1/2$ there is no successful competition in the precise sense defined in Sec. II. Successful competition is only possible for $\lambda < 1/3$ but, in any case, it is never very significant (only rather small fingers may be suppressed).

B. Comparison with the regularized dynamics

We are interested in the comparison between the $d_0=0$ dynamics and the $d_0 \neq 0$ one. The dynamical system defined by the mapping Eq. (5) is referred to as $L^2(\lambda)$. From now on we will restrict the analysis to the relevant case for $d_0 \to 0$, namely, $\lambda = 1/2$. In order to compare with the $d_0 \neq 0$ dynamics we first have to define an appropriate invariant manifold of the full dynamical system $S^\infty(d_0)$. Following Ref. [30] we can take a uniparametric set of initial conditions of the form Eq. (5) in a neighborhood of the PI fixed point, say $\alpha(\theta) = \varepsilon e^{i\theta}$ and define a two-dimensional manifold as the set of trajectories generated by the forward and backward evolution of those initial conditions with the dynamics of finite d_0. The resulting DS, which we call $S^2(d_0)$, is thus defined on a two-dimensional invariant manifold $\mathcal{S}^2(d_0)$ of the infinite-dimensional phase space of $S^\infty(d_0)$. That manifold intersects the one where $L^2(1/2)$ is defined, denoted by $\mathcal{L}^2(1/2)$ at the line of initial conditions parametrized by θ above and at PI. By taking the limit $\varepsilon \to 0$ then the two manifolds become tangent at PI. The basic conclusion of Ref. [30] was that the flow defined by the above DS's $L^2(1/2)$ and $S^2(d_0)$ are *not* topologically equivalent, in connection with the fact that $L^2(1/2)$ is structurally unstable. Accordingly, a generic perturbation of the equations, for instance, the one provided by the introduction of a small surface tension, does yield a qualitatively different system. In this sense, the DS's defined by $L^2(1/2)$ in no way can be the limit of the regularized system $S^2(d_0)$ as $d_0 \to 0$ since topological inequivalence means that there is no continuous deformation connecting the two phase portraits. Notice, however, that the manifold $\mathcal{S}^2(d_0)$ where $S^2(d_0)$ is defined is a different subset of the whole infinite-dimensional phase space for each value of d_0, all of them tangent at PI. This means that we are actually comparing interface configurations that are qualitatively similar but not quite the same. In order to strengthen the result, it is thus interesting to consider the limit $d_0 \to 0$, as proposed in Ref. [30]. By doing this we will guarantee that the regularized dynamics will converge to the zero-surface tension dynamics in some parts of phase space, namely, the trajectories connecting the PI fixed point respectively to the 1ST and the 2ST fixed points (selection theory does guarantee that, for $\lambda = 1/2$ $1\mathrm{ST}' \to 1\mathrm{ST}$ and $2\mathrm{ST}' \to 2\mathrm{ST}$). Within the framework of the singular perturbative analysis of Refs. [28,29] it is now clear that the regularized dynamics will converge to the idealized one in a finite (nonzero measure) region of $L^2(1/2)$, which includes the three fixed points and a neighborhood of the trajectories connecting them (the region defined by the zero surface tension dynamics until the impact at finite time on the unit circle of the so-called daughter singularities). Then the statement of the fundamental difference between the regularized and the idealized problems takes a stronger form in that the two respective manifolds coincide at order one time but depart from each other for the long-time dynamics that defines finger competition. Knowing the regions where the two manifolds coincide does unambiguously define the part of the dynamics that is correctly captured by the zero surface tension problem. Only for this part, introducing now a small but finite surface tension will behave as a regular perturbation. Hence although taking the limit of vanishing surface tension is not necessary to state the qualitative differences between the problem with and without surface tension, it clarifies and strengthens the conclusion on a quantitative basis. A detailed numerical study of this problem will be presented elsewhere [38]. At this point, a word of caution is required concerning the distinction between intrinsic dynamics and noise effects when the limit of very small surface tension is considered. The well-known sensitivity to noise of the ST solution when surface tension is decreased in the presence of noise [41] may modify in practice the present scenario making it virtually impossible for the dynamics to actually attain the fixed points [26]. It is important to stress, however, that while this is true for a fixed amount of local (high wave number) noise, either numerical or experimental, this effect is not contained in the intrinsic dynamics. That is, careful numerical studies have shown that the small surface tension limit can be approached to arbitrarily small values, provided that numerically generated noise is properly controlled [28,29,38,42]. Furthermore, it has been conclusively shown that, in the absence of noise, the single-finger fixed point is the universal attractor of the problem, at least for the classes of initial conditions considered here.

The flow topology of the regularized problem is thus very simple. PI is an unstable fixed point, $1\mathrm{ST}'$ is a stable fixed point, and $2\mathrm{ST}'$ is a saddle point with a stable manifold connected to PI and an unstable manifold connected to $1\mathrm{ST}'$. The model $L^2(1/2)$ instead, contains, in addition to PI, 1ST, and 2ST, an additional saddle fixed point that separates the basin of attraction of 1ST from the rest of the flow, which ends up in a continuum of fixed points corresponding to two unequal fingers. It is precisely the existence of this line of fixed points that causes the structural instability of the flow of $L^2(\lambda)$ according to Peixoto's theorem [39]. This is also responsible for the fact that the double finger 2ST fixed point is nonhyperbolic, that is, it misses the unstable direction that should connect 2ST to 1ST. From a physical point of view, it is clear that the saddle-point structure of the 2ST fixed point is essential to account correctly for finger competition, since it is the instability of this equal-finger configuration to symmetry-breaking perturbations that originates the phenomenon of finger competition. In this sense, we can associate "growth" with the stable direction of 2ST and "competition" with the unstable one. This saddle-point structure of the 2ST fixed point is thus expected to govern the crossover between these two regimes introduced above. In the following sections we will see that the failure of the minimal model $L^2(1/2)$ to properly account for finger competition is a generic property of the zero surface tension problem.

IV. EXTENSION WITHIN TWO DIMENSIONS: SEARCH FOR AN UNFOLDING

A. Modified minimal model

While the natural unfolding of the structurally unstable system is provided by surface tension, it would be desirable to find an unfolding of it within the class of integrable mappings with zero surface tension. In this way there would be hope of having a qualitatively correct description of finger competition. A possible modification of the ansatz (5) that is solvable and preserves the two dimensionality of the phase space is the following:

$$f(\omega,t)=-\ln\omega+d(t)+(1-\lambda+i\epsilon)\ln[1-\alpha(t)\omega]$$
$$+(1-\lambda-i\epsilon)\ln[1+\alpha(t)^*\omega], \qquad (6)$$

where ϵ is a real positive and is a constant of motion. Solutions of this type have been studied before, for instance, in Ref. [43]. This mapping describes generically two unequal axisymmetric fingers, with the symmetry axis located in fixed channel positions separated a distance π, half the channel width. The main morphological difference between the interfaces described by the minimal class Eq. (5) and those obtained from Eq. (6) is that the latter may present overhangs (see the detailed geometrical interpretation of parameters in Ref. [33]). An example of these solutions is shown in Fig. 1, with a series of snapshots of the corresponding time evolution. The class of solutions Eq. (6) contains also the single finger Saffman-Taylor solution $(\alpha'=0)$ but, remarkably enough, the introduction of a finite ϵ has removed the 2ST finger solution. The constant of motion λ is again the asymptotic width of the advancing finger. The natural phase space in this case is the unit circle, $|\alpha|\leqslant 1$, but we will restrict the study to $\alpha'\geqslant 0$ because the $\alpha'\leqslant 0$ region can be obtained by a π rotation of the $\alpha'\geqslant 0$ region. Physically, this rotation or the replacement $\alpha\to-\alpha$ corresponds to a shift of the interface by an amount π (half the channel width) in the y direction.

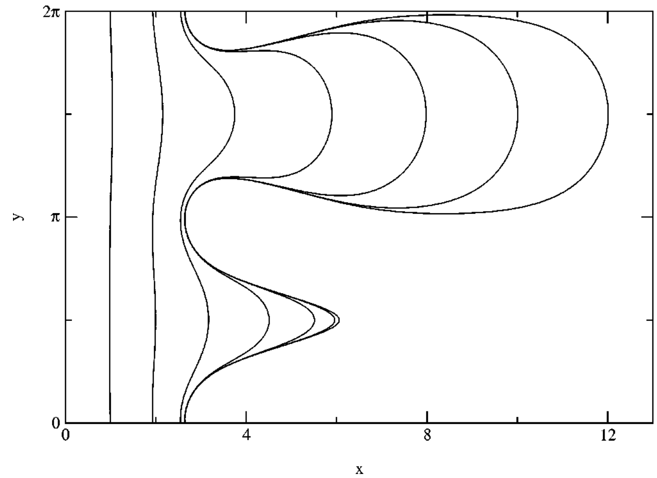

FIG. 1. Time evolution of a configuration with $\lambda=1/2$ and $\epsilon=0.1$.

For the minimal model the zeros ω_0 of $\partial_\omega f(\omega,t)$ laid outside the unit circle, but for the modified minimal model Eq. (6) the situation is different. For $|\alpha|<1$ a zero of $\partial_\omega f(\omega,t)$ can be inside the unit circle. It can be shown that for any λ and $\epsilon\neq 0$ a ω_0 can be found such that $|\omega_0|<1$ for some $|\alpha|<1$. For instance, with $\lambda=1/2$ the curve $|\omega_0(\alpha)|=1$ is the line $\alpha''=-1+2\epsilon\alpha'$ that clearly intersects the unit circle $|\alpha|=1$, enclosing a region where $|\omega_0|<1$. As a consequence of the presence of a zero inside the unit circle the parameter space $|\alpha|\leqslant 1$ contains unphysical regions, where the mapping Eq. (6) describes physically unacceptable situations, with self-intersection of the interface associated with the fact that the mapping is not single valued. One of these regions is defined by the existence of a zero ω_0 of $\partial_\omega f(\omega,t)$ inside the unit circle. In this region of phase space the interface crosses itself at one point, describing a single loop. Most remarkably, a second unphysical region containing interfaces with two intersections cannot be so easily detected since, in this case, the zeros of $\partial_\omega f(\omega,t)$ lay outside the unit circle. Zero surface tension solutions displaying this feature were also reported in Ref. [34]. Figure 2 shows a configuration with this double crossing.

The dynamical system defined by the ansatz Eq. (6) when inserted in Eq. (3) will be denoted by $L^2(\lambda,\epsilon)$ and the corresponding two-dimensional manifold $\mathcal{L}^2(\lambda,\epsilon)$. This DS can be integrated explicitly and the corresponding solutions for the variables $d(t)$ and $\alpha(t)=\alpha'(t)+i\alpha''(t)$ take the form

$$\beta=d(t)-\ln\alpha(t)+(1-\lambda-i\epsilon)\ln[1-|\alpha(t)|^2]$$
$$+(1-\lambda+i\epsilon)\ln[1+\alpha(t)^2], \qquad (7)$$

$$t+C=\lambda d(t)+(1-\lambda)\ln|\alpha(t)|-\epsilon\arctan\frac{\alpha''(t)}{\alpha'(t)}, \qquad (8)$$

where C is a real-valued constant and β is a complex-valued constant.

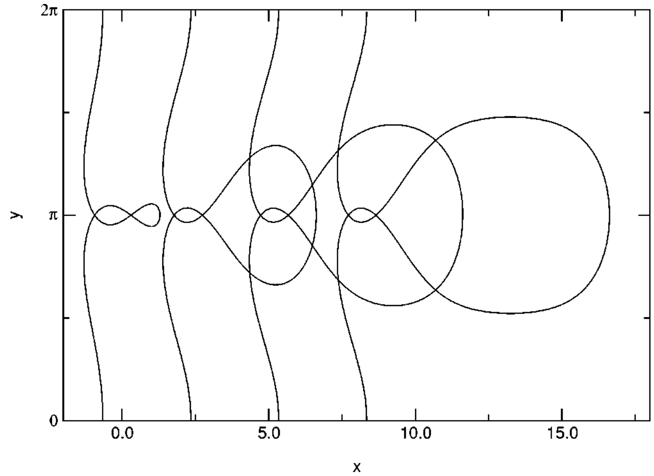

FIG. 2. Time evolution of a configuration with a double crossing of the interface, with $\lambda=\frac{1}{2}$ and $\epsilon=\frac{1}{2}$. The leftmost line corresponds to $t=0$ with $\alpha=0.85+i0.4$ and the rightmost line to $t=3.0$. (The curves are plotted with their mean x position shifted arbitrarily for better visualization.)

B. Study of the dynamical system

As depicted in Fig. 3, the introduction of an imaginary part $i\epsilon$ to the constant $(1-\lambda)$ modifies qualitatively the phase portrait of the minimal model, as expected from its structural instability [notice that a change in the initial condition for the mapping $f(\omega)$ takes the form here of a change in the form of the ODE's defining the DS]. Unfortunately, the phase portrait thus obtained does not have the structure of a saddle-point connection between the unstable and the stable fixed point, as would correspond to the natural unfolding provided by surface tension regularization. The phase portrait for $d_0\neq 0$ would be similar to that of $\epsilon=0$ in Fig. 3(a), except that the continuum of fixed points is no longer present and all trajectories other than the line $\alpha''=0$ would end up symmetrical to the upper ST fixed point or the lower one. Notice that in this representation, the 1ST fixed point has been split into two—1ST(R) and 1ST(L)—corresponding to whether the right or the left finger approaches the single finger attractor. These two solutions correspond to having the ST finger located at two different positions (the symmetry axes of the fingers) owing to the translational invariance associated with the periodic boundary conditions. This degeneracy of the attracting fixed points is only apparent, since the two points must be topologically identified as the same. This will in turn allow comparison with the case rigid-wall boundary conditions (see a detailed discussion in Sec. VI D).

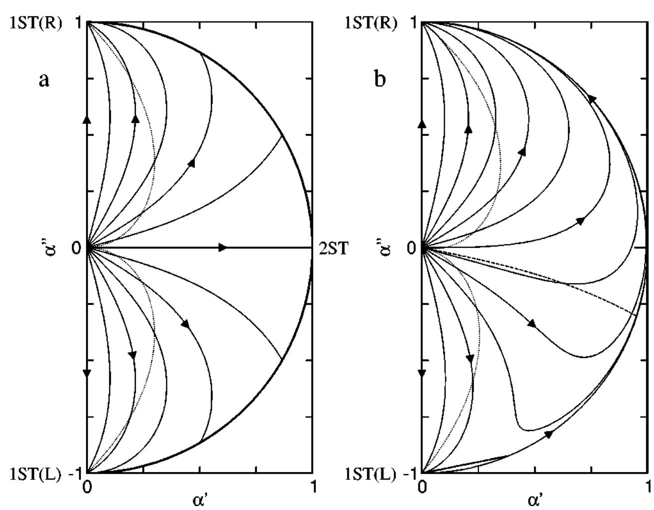

FIG. 3. Phase portrait of the minimal model and the modified minimal model. $\lambda=\frac{1}{2}$ for both plots, the regions to the right of the dotted lines correspond to two-finger configurations (a) $\epsilon=0$; note the continuum of fixed points (marked with a thick line) on $|\alpha|=1$. (b) $\epsilon=0.1$; the straight line in the lower left corner is a line of finite-time singularities and the two fingers have equal length on the dashed line.

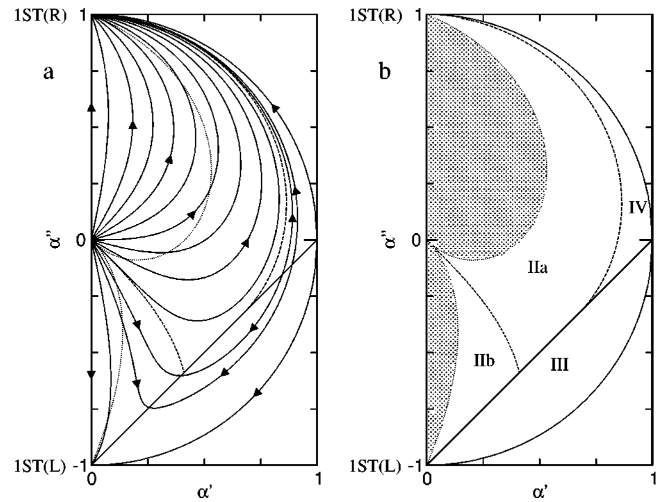

FIG. 4. (a) Phase portrait of the modified minimal model with $\lambda=1/2$ and $\epsilon=1/2$. (b) Plot of different regions of phase space of case (a). The gray regions correspond to single finger interfaces and the other regions to two finger interfaces. Regions IIa and IIb differ in which of the two fingers is larger. Regions III and IV are unphysical regions described in the text. The straight boundary of region III is a line of cusp singularities.

Therefore, we must conclude that the modified minimal model does not provide the correct unfolding. This is particularly remarkable if one takes into account that, in two-dimensional systems, structurally stable dynamical systems are dense [39]. On the contrary, the perturbed equations contain finite-time singularities and, although they remove the continuum of double-finger fixed points, they also miss the equal-finger fixed point, which is an essential ingredient of the regularized flow.

In Fig. 4 we plot the phase portrait for $\epsilon=0.5$ and the different regions of phase space. For any other ϵ the flow is topologically equivalent but the shape and size of the different regions vary smoothly. The line of finite-time singularities collapses towards the lower fixed point 1ST(L) in the limit $\epsilon\to 0$ as shown in Fig. 3(b). Because of the absence of the 2ST fixed point, the splitting of flow is made possible by the existence of the line of finite-time singularities. Instead of a separatrix between the respective basins of attraction of 1ST(R) and 1ST(L), there is an intermediate, nonzero measure region, connected to the PI fixed point, whose evolution ends up at that singularity line, defined by the condition

$|\omega_0|=1$. Similarly to the finite-time singularities occurring for polynomial mappings, this line is reached in a finite time and is associated with the formation of a cusp at the interface. The evolution is not defined after that time. The flow in the region below the singularity line [region III of Fig. 4(b)], defined by $|\omega_0|<1$, is actually well defined although it describes evolution of unphysical interfaces that intersect themselves forming a loop. Their evolution originates and ends at different points of the singularity line. The region IV has double crossings of the interface (see Fig. 2) and also originates at the singularity line but, remarkably enough, it evolves asymptotically towards the ST finger despite their unphysical double crossing at the tail of the finger. This double crossing is removed in a finite time in some subregion of IV and it remains up to infinite time in another subregion. This clearly illustrates the necessary caution when drawing conclusions on the dynamics from the fact that the interface evolves asymptotically towards a single ST finger. In fact, with zero surface tension dynamics smooth and apparently physical interfaces may contain elements that yield them physically unacceptable when the time evolution is considered either forward or backward, even without involving cusp formation.

Incidentally, the double-crossing removal in some of the above solutions has some implications in the general study of topological singularities associated with interface pinchoff in fluid systems. Consider the stable Saffman-Taylor problem, in which the viscous fluid displaces the inviscid one. The planar interface is stable in this case and is the attractor of the dynamics. The conformal mapping obeys the same evolution equation Eq. (3), except for time reversal $t\rightarrow -t$. As a consequence, the double-crossing removal we observe in our setup encompasses a prediction of a finite-time interface pinchoff in the stable configuration of the problem, for some class of initial conditions. A similar pinchoff phenomenon for zero surface tension dynamics was detected numerically by Baker, Siegel, and Tanveer [34] for other types of mappings. Our result provides a very simple example of an exactly solvable finite-time pinchoff. Notice that there is no singularity of the interface shape or velocity at the interface contact, so one could presume that surface tension may not significantly affect the phenomenon in this case, although this is an open question yet.

Disregarding the time direction, the graph $\alpha''(\alpha')$ for the modified minimal model is continuous and differentiable in all regions including the unphysical region III. With the definition $\alpha=re^{i\theta}$, Eq. (7) yields, after some algebra,

$$\frac{d\theta}{dr}=\frac{4r\cos\theta}{1-r^2}\times\frac{(1-\lambda)(1-r^2)\sin\theta+\epsilon(1+r^2)\cos\theta}{1+(2\lambda-1)r^4+2\lambda r^2\cos 2\theta+2\epsilon r^2\sin 2\theta}. \quad (9)$$

The fact that the modified minimal model does not yield an unfolding of the minimal one is more deeply stressed by the fact that the field of directions defined by the above graph, even after removing the singularities through a proper time reparametrization and after time reversal in region III, is still a structurally unstable flow.

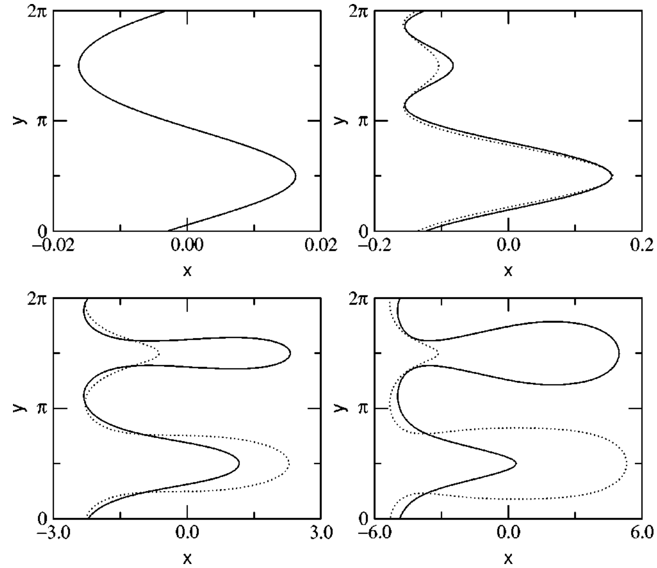

FIG. 5. Evolution of two interfaces initially equal to linear order (see text), with $\lambda=1/3$ and $\epsilon=0.1$. $\alpha(0)=0.046\,193\,98-i0.019\,134\,17$ for the solid line and $\alpha(0)=-0.046\,193\,98-i0.005\,275\,98$ for the dashed line. Upper left plot, $t=0$; upper right plot, $t=2.0$; lower left plot, $t=4.0$; and lower right plot, $t=6.0$.

The ill posedness of the zero surface tension case as an initial-value problem [27] manifests in that arbitrarily close initial conditions may differ dramatically after a finite time. For instance, a polynomial mapping will always develop a finite-time cusp but can be as close as desired to any initial condition that will remain smooth for all time. In the following we briefly describe some illustrative examples of such sensitivity to initial conditions in much less foreseeable situations.

(a) Example 1. Consider two initial conditions (α_1',α_1'') and (α_2',α_2'') close to the PI fixed point, with $|\alpha_1|,|\alpha_2|\ll 1$, which differ only in nonlinear orders of their mode amplitudes [44]. One can easily choose (α_2',α_2'') (with $\alpha_1'\alpha_2'<0$, that is, considering not only the semicircle $\alpha'>0$ but the whole unit circle) such that the time evolution will be completely different from the evolution of the original initial condition, even though the two initial conditions were equivalent to linear order. In Fig. 5 we show an explicit example. While the two initial conditions for the interface configuration cannot be distinguished in the scale of the plot, the final outcome is dramatically different. One of the evolutions is an example of successful competition, where the finger in the initial condition is eventually approaching the ST solution, with a small secondary finger (not present in the initial condition) that is generated but screened out later on by the leading one. The other evolution is quite surprising since the secondary finger grows to the point of taking over and winning the competition.

(b) Example 2. A similar situation is found if one compares two initial conditions equivalent to linear order up to a

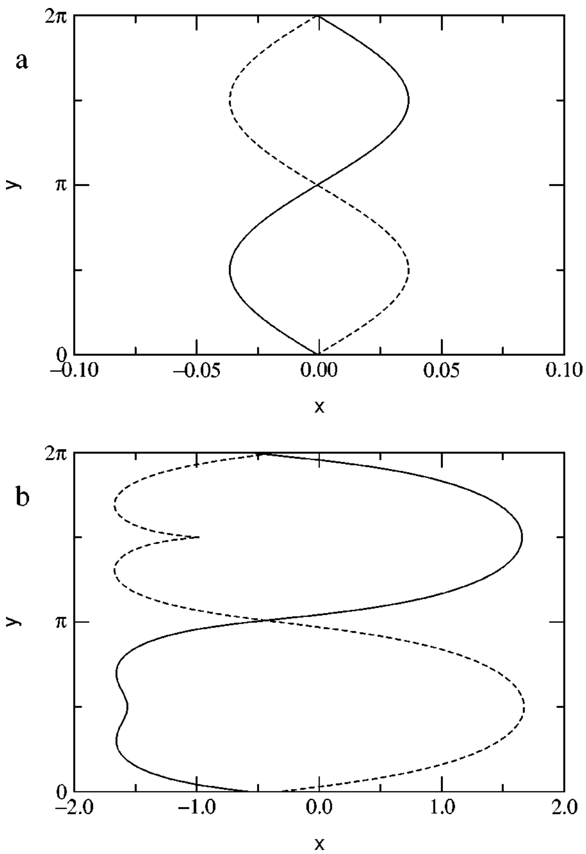

FIG. 6. Evolution of two interfaces symmetric to linear order (see text), with $\lambda=\frac{1}{2}$, $\epsilon=0.1$, $\alpha(0)=0.027\,24+i0.031\,04$ for the solid line and $\alpha(0)=0.027\,24-i0.041\,93$ for the dashed line. The upper plot corresponds to $t=0$ and the lower to $t=4.19$, when a cusp develops.

parity transformation. Pairs of initial conditions of this type, with the same values of λ and ϵ, can easily be found within the same semicircular phase space, and since the dynamics is indeed symmetric under mirror reflection, naively one would not expect a very different behavior, even though such points are not close to each other in phase space. Figure 6 shows an example in which one of the evolutions is smooth, with a leading finger and a small one being generated, and the other generates a cusp in finite time. As in the first example, no signature of the different fate of the system could apparently be seen in the initial conditions. In both cases the extremely small differences associated with higher orders in the mode amplitudes have thus been crucial. The sensitivity to initial conditions of these examples is more striking for decreasing values of ϵ, since the time in which the two evolutions stay close to each other increases as $O(-\ln\epsilon)$. For instance, given an initial condition α_0 close to PI, the difference between the $\epsilon=0$ interface and the $\epsilon\to 0$ one will remain of $O(\epsilon)$ for a time of $O(-\ln\epsilon)$. Later on in the evolution the differences between the two interfaces will be of $O(1)$: the asymptotic shape of the $\epsilon=0$ case will be two unequal fingers while the shape of the $\epsilon\to 0$ will be a single Saffman-Taylor finger. Similarly, for two initial conditions symmetrical to linear order such as in example 2, with $\epsilon\to 0$, the differences between their interfaces will remain symmetric to $O(\epsilon)$ for a time of $O(-\ln\epsilon)$, but later they will lose symmetry and finally both will end up at the same fixed point, say, the right one, even though one of the two evolutions has been favoring the other one, say, the left one, for a long time (up to well-developed fingers). Similarly, the evolution of initial conditions that are identical to linear order but that have different ϵ may be dramatically different.

The above examples clearly call for caution when trying to use exact solutions as approximants of the full (regularized) dynamics of the problem. A direct comparison of these solutions with numerical integration for very small surface tension would be required in order to make a more quantitative assessment of this point. This will be presented elsewhere [38]. In any case, it must also be stated that the class of logarithmic solutions does provide also qualitatively correct evolutions, not only of single-finger configurations as stated in Ref. [30], but also with two-finger configurations showing successful competition. An example of this is plotted in Fig. 1. Starting from the planar interface, during the linear regime a bump starts to grow, followed generically by a second bump as the evolution enters the nonlinear regime. The two fingers keep on growing for some time, until one of them is suddenly eliminated from the competition as the other finger approaches asymptotically the ST finger solution. This general scenario is illustrated in Fig. 7(a), where the individual growth rates of the two fingers $\Delta\psi_1$ and $\Delta\psi_2$ are plotted versus time, for two different initial conditions.

For other initial conditions as generic as the previous one, however, anomalous competition is observed, in the sense that the finger suppressed is the larger one. An example of this phenomenon is shown in Fig. 7(b) where, initially, only one finger has a finite $\Delta\psi_1$. This grows for a while but eventually a second finger develops and begins to grow, as indicated by the appearance of a nonzero $\Delta\psi_2$. The second finger's growth rate increases faster and the finger surpasses the first one, which becomes finally suppressed. This is indicated by $\Delta\psi_1$ going to zero. This is an interesting example where there is successful competition (finger coalescence) to the Saffman-Taylor asymptotic solution but with a presumably wrong dynamics in comparison with the regularized problem. In fact it can be seen that the zero surface tension evolution departs from the regularized trajectory much before the small finger takes over the competition (through the impact of a daughter singularity [28]). The winning finger with the regularized dynamics is thus the losing finger with the zero surface tension one [38].

Again, in the limit $\epsilon\to 0$ these phenomena appear even more dramatically, as a consequence of the structural instability of the minimal model. In this limit, for a $O(-\ln\epsilon)$ time we will observe two unequal fully developed fingers advancing with a fixed tip distance, but eventually the presence of finite ϵ will "activate" the competition process and one of the two fingers will reduce its growth rate until fully suppressed from the competition. If $\alpha''(0)>0$ the suppressed finger will be the small one, but if $\alpha''(0)<0$ the dynamically suppressed finger will be the large one.

C. Comparison with the regularized dynamics

In order to compare the $d_0=0$ dynamics with the physical case of $d_0\neq 0$, we use the construction introduced in Sec.

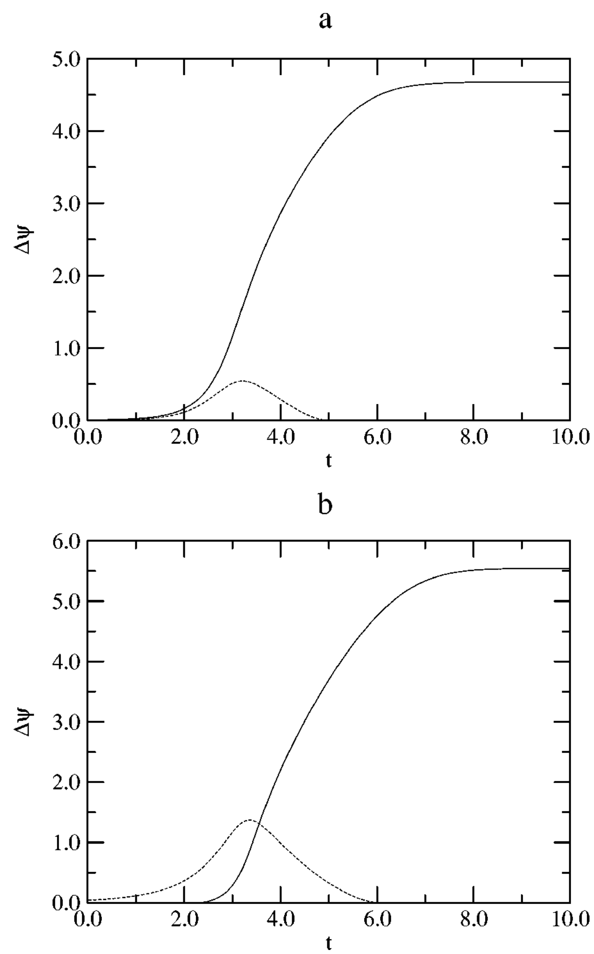

FIG. 7. Individual growth rates $\Delta\psi_1(t)$ and $\Delta\psi_2(t)$ of the two fingers for the modified minimal model with $\lambda=\frac{1}{2}$ and $\epsilon=0.1$, for two different initial conditions showing successful competition. For the (a) case the finger that initially has larger growth rate (and larger length too) wins the competition. For the (b) case the finger that initially has lower growth rate (and lower length too) wins the competition, in opposition to the evolution with the regularized dynamics (small surface tension).

III B. This defines a two-dimensional dynamical system $S^2(d_0)$ on a surface $\mathcal{S}^2(d_0)$ that is tangent, by construction, to the zero surface tension counterpart $\mathcal{L}^2(1/2,\epsilon)$ at the PI fixed point. We can also define the limiting case $S^2(0^+)$ as the limit of $S^2(d_0)$ for $d_0\to 0$. From the results of Ref. [29] it follows that $\mathcal{L}^2(1/2,\epsilon)$ and $\mathcal{S}^2(0^+)$ intersect not only at the 1ST(R) and 1ST(L) (selection theory) but have in common the full evolution of the $d_0=0$ time-dependent single-finger solution (line $\alpha'=0$). For the set of dynamical systems $S^2(d_0)$ defined for different values of d_0 the basins of attraction of 1ST(R) and 1ST(L) are two-dimensional and finite, and therefore there must be at least one separatrix trajectory between the two basins. This separatrix must end at a saddle fixed point (which does not exist in the phase portrait of the $d_0=0$ solution). It is reasonable to assume that this fixed point is the double ST finger fixed point (2ST′). Thus, the topology of the flow defined by the dynamical system with $d_0=0$, $L^2(1/2,\epsilon)$ is not equivalent to the flow of the dynamical system as $d_0\to 0$, $S^2(0^+)$: the flow for the regularized problem contains a trajectory and a fixed point that it is not contained in the flow defined by the modified minimal model, the trajectory starting at the planar interface PI fixed point and ending up at the 2ST fixed point. The phase flow of the modified minimal model with $d_0=0$ is qualitatively different from the phase flow of the regularized problem, $d_0\to 0$, and therefore the solution Eq. (6) is unphysical in a global sense, what is to say, when a sufficiently large set of initial conditions [spanning evolutions towards 1ST(R) and 1ST(L)] is considered simultaneously. Again it is important to state that the strict limit $d_0\to 0$ is not necessary in order to reach our basic conclusion on the topological inequivalence of the regularized and the idealized systems. The limit is taken to emphasize that the manifold $\mathcal{S}^2(d_0)$ is indeed close to $\mathcal{L}^2(1/2,\epsilon)$ and subsets of it do converge to $\mathcal{L}^2(1/2,\epsilon)$ (see discussion in Sec. III B).

We have shown that the introduction of a finite $i\epsilon$ term into the minimal model Eq. (5) fails to provide an unfolding of its nonhyperbolic fixed-point structure. It has dramatically changed the topology of the flow obtained for $\epsilon=0$, but the flow for $\epsilon\neq 0$ does not have the expected structurally stable flow of the physical problem (for two-finger configurations): an unstable fixed point, two stable fixed points, and one saddle fixed point. Moreover, instead of this, the evolution of Eq. (6) with $\epsilon\neq 0$ presents finite-time singularities for a nonzero measure set of initial conditions. This can be understood as a consequence of the absence of the 2ST saddle point, which should control the competition regime. Without this fixed point the separatrix trajectory between the basins of attraction of ST(L) and ST(R) is not present and the only possible way to split the flow is through the existence of finite time singularities. This is not a particularity of the mapping Eq. (6) but a more general feature of $d_0=0$ solutions. Below we will prove that, within the N-logarithms class, finite ϵ implies finite-time singularities in the evolution of a nonzero measure set of initial conditions (see Sec. V C). Besides the existence of finite-time singularities we have seen that, unlike the case $\epsilon=0$, solutions exhibiting successful competition are possible with $\epsilon\neq 0$ for $\lambda=1/2$. However, part of those evolutions are unphysical in the sense the winning finger may differ from the one with the regularized dynamics.

V. GENERALIZATION TO HIGHER DIMENSIONS

This section is devoted to the study of solutions that define a dynamical system of higher dimension and less symmetry. We will show that the conclusions of previous sections do apply in a much more general setting.

A. Nonaxisymmetric fingers

The solutions that have been studied in the previous sections, Eqs. (5) and (6), have two polelike singularities $\omega_{1,2}$ located at $\omega_1=1/\alpha$ and $\omega_2=-1/\alpha^*$. The property $\omega_1=-\omega_2^*$ reduces the dimensionality of the dynamical system to two and also forces the axisymmetry of the fingers. If the singularities $\omega_{1,2}$ are completely arbitrary, then the phase space has one additional dimension and the fingers are not axisymmetric. The ansatz

$$f(\omega,t)=-\ln\omega+d(t)+(1-\lambda)\ln[1-\alpha_1(t)\omega]+(1-\lambda)\ln[1-\alpha_2(t)\omega] \tag{10}$$

is solvable and is free of finite-time singularities. It has been studied in detail in Ref. [4], where it has been proven that the two-dimensional, axisymmetric case (minimal model) is always attractive with respect to this departure from axisymmetry, that is the three-dimensional phase portrait corresponding to solutions of the form Eq. (10) converges asymptotically to that of the minimal model. Therefore, the conclusions from the minimal model are robust to such symmetry-breaking perturbations.

Similarly, other symmetry-breaking perturbations that do not increase the dimensionality are described by integrable maps of the form

$$f(\omega,t)=-\ln\omega+d(t)+(1-\lambda+p+i\epsilon)\ln[1-\alpha(t)\omega]+(1-\lambda-p-i\epsilon)\ln[1+\alpha^*(t)\omega], \tag{11}$$

where $0<p<1-\lambda$. In the case $p\neq 0$, however, the phase portraits obtained for $p=\epsilon=0$, are not qualitatively modified. The continuum of fixed points present for $\epsilon=0$ is not removed by the introduction of a finite p and the finite-time singularities that appear for $\epsilon\neq 0$ are also present when $p\neq 0$. Therefore, we conclude that breaking the symmetry does not modify the general scenario discussed in previous sections.

B. Perturbations which change finger widths

Consider now a modification of the ansatz (5) of the form

$$f(\omega,t)=-\ln\omega+d(t)+(1-\lambda)\ln[1-\alpha_1(t)\omega]+(1-\lambda)\ln[1-\alpha_2(t)\omega]+2(\lambda-\lambda_s)\ln[1-\delta(t)\omega] \tag{12}$$

with initial conditions $\alpha_1(0)=-\alpha_2^*(0)=\alpha(0)$, $0<\lambda,\lambda_s<1$ and $|\delta(0)|\ll 1$. From substitution of this ansatz into the evolution equation (3) it is obtained that Eq. (12) is a solution with λ and λ_s constants. From the dynamical equations it can be proved that the asymptotic configuration of this ansatz consists of one or two fingers, with asymptotic filling fraction equal to λ_s. But if $|\delta(0)|\ll|\alpha(0)|$ then the interface will be initially almost identical to the one obtained within the class (10) with the same $\alpha(0)$ and λ, and its evolution will remain close to the one obtained for Eq. (10), for a time that will increase for decreasing $|\delta(0)|$. Therefore, given a small enough $|\delta(0)|$, a configuration with one or two fingers (depending on the initial conditions) of total width λ will develop. Later on, as $|\delta|$ grows and approaches 1, the total width will change from λ to λ_s for long enough time. The ansatz (12) thus describes an interface that changes the filling fraction of the fingers from λ to λ_s. The same phenomenon will appear with any other of the solutions described in this paper (and in general in polelike solutions) if a term of the type $2(\lambda-\lambda_s)\ln[1-\delta(t)\omega]$ is added. This changing-width phenomenon of $d_0=0$ solutions has been known for long [32], but it has been recently claimed [21] to dynamically explain finger width selection without the need to invoke surface tension. The idea was that, although solutions of arbitrary λ exist in the absence of surface tension, these are unstable under some perturbations that trigger the evolution towards the $\lambda=1/2$ solution. Since the present paper is basically emphasizing the unphysical dynamics of the idealized $(d_0=0)$ problem, in direct contradiction with Ref. [21], we feel compelled to briefly comment on this respect here. The basic argument of Ref. [21] is as follows, in terms of the parametrization of the interface used by the author: a term of the form $i\mu\phi$ in the conformal mapping is always unstable under the substitution $i\mu\phi\rightarrow\mu\ln(e^{i\phi}-\epsilon)$. The introduction of such perturbation then leads to the $\mu=0$ case, which corresponds to $\lambda=1/2$. In Refs. [22,23] it was pointed out that, with the same degree of generality, equivalent perturbations exist that lead to any desired λ, and therefore the conclusion that $\lambda=1/2$ is the only attractor is incorrect. It is argued [24] that the latter class of perturbations is different form the former since they increase the number of logarithmic terms in the conformal mapping and therefore modify the dimension of the subspace of solutions. This objection is somewhat misleading since such partitioning of classes of solutions in terms of the number of logarithms is arbitrary and not intrinsic. This can be seen by choosing a different reference region to conformally map the physical fluid. Instead of mapping it into the semi-infinite strip [21], the mapping into the interior of the unit circle avoids the confusion on the dimension of the subspace of solutions. Thus, the perturbation proposed in Ref. [21] is equivalent to choosing $\lambda_s=1/2$ in the ansatz (12), but it is manifest in this formulation that there is nothing special with this particular choice of λ_s. Perturbations leading to any finger width λ_s occur with the same generic nature. Therefore, the instability of the point $\delta=0$ is *not* related to the steady-state selection phenomenon.

C. Finite-time singularities within *N*-logarithm solutions

In this section we will prove that solutions of the *N*-logarithm class [33] that do not have only real constant parameters contain nonzero measure sets of (smooth) initial conditions that develop finite-time singularities.

Consider a conformal mapping function $f(\omega,t)$,

$$f(\omega,t)=-\ln\omega+d(t)+(\Lambda_1+i\epsilon)\ln[1-\alpha_1(t)\omega]+(\Lambda_2-i\epsilon)\ln[1-\alpha_2(t)\omega], \tag{13}$$

where $\Lambda_1+\Lambda_2=2(1-\lambda)$, $\epsilon>0$ and $\alpha_{1,2}$ are complex with $|\alpha_{1,2}|<1$. The mapping $f(\omega,t)$ must satisfy $\partial_\omega f(\omega,t)\neq 0$ for $|\omega|\leqslant 1$. If any zero ω_0 of $\partial_\omega f(\omega,t)$ hits the unit circle $|\omega|=1$ then the interface develops a cusp. For the ansatz (13), $\partial_\omega f(\omega,t)$ reads

$$\partial_\omega f=-\frac{1}{\omega}-\frac{(\Lambda_1+i\epsilon)\alpha_1}{1-\alpha_1\omega}-\frac{(\Lambda_2-i\epsilon)\alpha_2}{1-\alpha_2\omega}. \tag{14}$$

Thus, the position of the zero ω_0 of $\partial_\omega f(\omega_0,t)$ is

$$\omega_0=\frac{-(\Lambda_1+i\epsilon-1)\alpha_1-(\Lambda_2-i\epsilon-1)\alpha_2}{2\alpha_1\alpha_2(2\lambda-1)}\pm\frac{\sqrt{((\Lambda_1+i\epsilon-1)\alpha_1+(\Lambda_2-i\epsilon-1)\alpha_2)^2-4\alpha_1\alpha_2(2\lambda-1)}}{2\alpha_1\alpha_2(2\lambda-1)}. \tag{15}$$

If, for some value of $\alpha_{1,2}$, $|\alpha_{1,2}|\leqslant 1$, the zero ω_0 is inside the unit circle, then the ansatz (13) will present finite-time singularities for some sets of initial conditions. Therefore, if $|\omega_0|<1$ the interface will develop a cusp. Setting $\alpha_{1,2}=\alpha e^{i\theta_{1,2}}$ and $\theta_2-\theta_1=-2\delta$ with $\delta\ll 1$, the position of the zero (keeping up to linear terms in δ) is

$$\omega_0=e^{-i\theta_2}\frac{\lambda\pm(1-\lambda)}{\alpha(2\lambda-1)}+\frac{i\delta e^{-i\theta_2}}{\alpha(2\lambda-1)}\left[\Lambda_2-1-i\epsilon\pm\frac{\lambda-1+\lambda(\Lambda_2-i\epsilon)}{1-\lambda}\right]+O(\delta^2) \tag{16}$$

and the modulus of the minus solution (the one with smaller modulus) reads

$$|\omega_0|=\frac{1}{\alpha}\left[1-\frac{\epsilon\delta}{1-\lambda}+O(\delta^2)\right]. \tag{17}$$

In consequence, for α close to 1 we obtain $|\omega_0|<1$, one of the zeros is inside the unit circle in a finite neighborhood of $\alpha_1=\alpha_2=e^{i\theta}$. Thus, the mapping (13) presents finite-time singularities for some initial conditions independently of the value of ϵ and $\Lambda_{1,2}$, and the measure of this set is nonzero.

Now we consider a generic mapping with $N>2$ logarithmic terms of the form

$$f(\omega,t)=-\ln\omega+d(t)+\sum_{j=1}^{N}\gamma_j\ln[1-\alpha_j(t)\omega], \tag{18}$$

where $\gamma_j=\Lambda_j+i\Gamma_j$ are constants of motion with the restriction $\Sigma_{j=1}^{N}\gamma_j=2(1-\lambda)$. If we choose $\alpha_j=\alpha_1$ for $1\leqslant j\leqslant k$ and $\alpha_j=\alpha_2$ for $k+1\leqslant j\leqslant N$, we recover the mapping (13). Therefore, the N-logarithm solution (18) contains initial conditions that develop a cusp with this subset of α_j, but the dimension of this subset is lower than the dimension of the phase space, implying that the measure of this subset would be zero. To prove that the subset of initial conditions that develops cusps has finite measure we choose now the following values for α_j: $\alpha_j=\alpha_1+\eta_j$ for $1\leqslant j\leqslant k$ and $\alpha_j=\alpha_2+\eta_j$ for $k+1\leqslant j\leqslant N$, with $|\eta_j|\ll 1$, where $|\omega_0|<1$ if $\eta_j=0$. The equation $\partial_\omega f(\omega,t)=0$ reads

$$\frac{1}{\omega}+\sum_{j=1}^{k}\frac{\gamma_j(\alpha_1+\eta_j)}{1-(\alpha_1+\eta_j)\omega}+\sum_{j=k+1}^{N}\frac{\gamma_j(\alpha_2+\eta_j)}{1-(\alpha_2+\eta_j)\omega}=0. \tag{19}$$

This equation (19) reduces to Eq. (15) if all $\eta_j=0$ and it has N zeros if $\eta_j\neq 0$. Defining $g(\omega)=\partial_\omega f(\omega,t)$ for $\eta_j=0$ and $G(\omega,\vec{\eta})=\partial_\omega f(\omega,t)$ for $\eta_j\neq 0$, then $G(\omega,\vec{\eta})=g(\omega)+\delta G(\omega,\vec{\eta})$, where $|\delta G(\omega,\vec{\eta})|<K|\vec{\eta}|$ for $|\omega|<R$, with K and R constants, and $g(\omega_0)=0$. One zero ω_0' of $G(\omega,\vec{\eta})$ can be written as $\omega_0'=\omega_0+\delta\omega$, and assuming $|\delta\omega|<C|\vec{\eta}|$ with C constant, the substitution of ω_0' in $G(\omega,\vec{\eta})=0$ yields

$$g(\omega_0)+\left.\frac{\partial g}{\partial\omega}\right|_{\omega_0}\delta\omega+\delta G(\omega_0,\vec{\eta})=0. \tag{20}$$

The position of the zero is then

$$\omega_0'=\omega_0-\frac{\delta G(\omega_0,\vec{\eta})}{\left.\frac{\partial g}{\partial\omega}\right|_{\omega_0}}, \tag{21}$$

where

$$\left.\frac{\partial g}{\partial\omega}\right|_{\omega_0}\neq 0.$$

Therefore, the zero ω_0' of Eq. (19) is inside a ball of radius $o(|\vec{\eta}|)$ centered in ω_0. If $|\omega_0|<1$, then choosing $|\vec{\eta}|$ small enough the zero will satisfy $|\omega_0'|<1$: in a neighborhood of (α_1,α_2) at least one zero of $\partial_\omega f(\omega,t)$ is inside the unit circle, and the dimension of this neighborhood will be the same as that of the phase space. So we can conclude that any mapping of the form (18) presents finite-time singularities for some sets of initial conditions of nonzero measure, provided that at least one pair of γ_j has a nonzero imaginary part.

Thus, the requirement that a mapping function of the form (18) is free from finite-time singularities for any initial condition $\alpha_j(0)$ is fulfilled if and only if $\mathrm{Im}[\gamma_j]=0$, $j=1,\ldots,N$. But this restriction implies [45] that for a wide range of initial conditions the asymptotic configuration is a N-finger interface with unequal fingers advancing at a constant speed, a situation fully analogous to the one discussed in Sec. III. Then, if a mapping of the form (18) with $\mathrm{Im}[\gamma_j]=0$ is chosen, the dynamical system $L^{2N}(\gamma_j)$ will have nonhyperbolic fixed points (continua of fixed points) and will lack the saddle-point structure of the regularized problem. In order to completely remove the continua of fixed points it is necessary to set $\mathrm{Im}[\gamma_j]\neq 0$ [45], but in this case we will encounter finite-time singularities and the saddle-point structure will not be present anyway.

To sum up, we have shown that the features of the minimal model and its extensions that make them globally unphysical are not specific to their low dimensionality or their symmetries. The features that make the solutions studied in previous sections ineligible as a physical description of small surface tension dynamics for a sufficiently large class of initial conditions (including finger competition), are also present within the much more general N-logarithm class of solutions, and the conclusions drawn in previous sections do apply to that general class.

D. Rigid-wall boundary conditions

It is worth stressing here that the use of periodic boundary conditions throughout this study, as opposed to the physically more natural rigid-wall boundary conditions, is not essential to the basic discussion. In connection with the discussion of multifinger steady solutions, this point was raised in Ref. [46] and addressed in Ref. [47]. Here we will just recall that the choice of periodic boundary conditions is not only the simplest in terms of symmetry and dimensionality, but it is the relevant one if one is interested in general mechanisms of finger competition in finger arrays. In this sense, the study of the two-finger configurations in this paper refers to an alternating mode of two-finger periodicity in an infinite array of fingers, in the spirit of Ref. [48]. For finite-size systems one can also argue that rigid-wall boundary conditions are included as a particular case of periodic boundary conditions in an enlarged system. That is, a channel with width W with rigid walls is mathematically equivalent to a channel of width $2W$ with periodic boundary conditions where auxiliary channel of width W is constructed as the mirror image of the physical one. The competition of two fingers in a channel with rigid walls at a distance W is in practice equivalent to a four-finger problem with periodic boundary conditions in a double channel.

The only subtle point that we would like to point out is the apparent degeneracy of the single-finger attractor into a left ST finger and a right ST finger, as already pointed out in Sec. IV B, and the possible relevance of this fact in connection with the saddle-point structure of the phase space flow. This degeneracy is inherited from the trivial continuous degeneracy associated with translation invariance in the transversal direction, when periodic boundary conditions are assumed. In fact an arbitrary shift in the transversal direction yields a physically equivalent configuration. When an initial condition is fixed, such continuous degeneracy is broken into two discrete spatial positions that are separated by a distance of $W/2$. The whole dynamical system is then invariant under translations of $W/2$. This is the reason why we only plotted a half of the disk in the phase portraits of Sec. IV. Technically, the resulting dynamical system must be defined ‘‘modulo-$W/2$,’’ that is, identifying any configuration with the resulting of a $W/2$ shift. In the phase space defined by the variables (α', α'') one should identify any point with its image under a π rotation. In this way the two single-finger attractors do correspond to the same fixed point. With this identification, the ST finger is not degenerate and the flow becomes topologically equivalent to the corresponding one in a channel with rigid-wall boundary conditions. The two-finger configurations have thus the same structure, regardless of the type of sidewall boundary conditions. The flow starts at the PI fixed point and ends up at the 1ST fixed point. Between them there is a saddle point corresponding to the 2ST fixed point. This separates the flow in two equivalent regions, namely, ‘‘from the left’’ and ‘‘from the right’’ of the saddle point. With zero-surface tension, the case of rigid walls exhibits the same problems, namely, the occurrence of a (nontrivial) continuum degeneracy of multifinger solutions, and the existence of finite-time singularities. The important point we want to stress is thus that all the general conclusions drawn in this paper are valid if rigid-wall boundary conditions are considered.

VI. DYNAMICAL SOLVABILITY. GENERAL DISCUSSION

A. The physics of zero surface tension

The role of the zero surface tension solutions in the description of the dynamics of the nonzero but vanishingly small surface tension problem is now clearer. The $d_0=0$ dynamics is in general incorrect in a global sense, even if we choose solutions with the asymptotic width λ given by selection theory. However, they have an important place in the description of the physical problem. It has been proved in Refs. [27–29] that the solutions with $d_0=0$ converge to the $d_0\to 0$ during a time $O(1)$, before the impact with the unit circle of the so-called *daughter singularity* at time t_d. In practice this implies that the $d_0=0$ dynamics is not only correct (with d_0 acting as a regular perturbation) in the linear regime but also quite deep into the nonlinear regime. After t_d nothing can be said *a priori*: as we have shown in the present paper, there are regions of the $d_0=0$ phase space corresponding to smooth interfaces with physically wrong dynamics, but other regions are a good description of the evolution with small but finite surface tension. For instance, in the neighborhood of the time-dependent Saffman-Taylor finger [the line $\alpha'=0$ in the solutions (5), (6)] the $d_0=0$ evolution is qualitatively correct for finite surface tension, and even quantitatively correct in the limit $d_0\to 0$ (for $\lambda=1/2$). However, a question remains open: given a $d_0=0$ evolution smooth for all time and consistent with the results of selection theory, is it the limit of a $d_0\to 0$ evolution? This question can be explored numerically and is the subject of a forthcoming paper [38]. Generally speaking, the conclusion is that exact solutions including evolution of two different fingers that are compatible with MS theory, that is, evolving to a single finger with the width predicted by selection theory, and that do not exhibit any kind of singularity in the interface shape, may be dramatically affected by surface tension. The outcome of the competition (that is, which one of the two competing fingers will survive at the end) when an infinitesimally small surface tension in introduced, may be the opposite one to that of the zero surface tension case. This may happen in situations where fingers are significantly different from each other and is not an instability of a particular trajectory, but a generic behavior in a finite (nonzero measure) range of initial conditions within the integrable class. For that region of phase space, it is clear that the dynamics of finger competition is completely wrong for the class of integrable solutions. Nevertheless, there is also a class of initial conditions that have a qualitatively correct evolution including ‘‘successful’’ finger competition in the sense defined in sections above (this possibility was incorrectly excluded in Ref. [30], where the analysis was based on $\epsilon=0$). Although strict convergence of the regularized solution to the idealized one may not occur in these cases, the quantitative differences may be moderately small. Actual convergence of some type can only be expected at most when there is only one finger along the complete time evolution.

In summary, there are basically four classes of initial conditions within the most general integrable solutions, once those *a priori* incompatible with selection theory are excluded, namely, (i) finite-time singularities forward or backward (or both) in time; (ii) asymptotically correct ST finger with wrong dynamics (the incorrect finger wins); (iii) asymptotically correct ST finger with qualitatively correct evolution (the correct finger wins although shapes may differ during a transient); and (iv) (unphysical) evolution towards multifinger fixed points. It has to be added that, all of the above solutions plus those that are incompatible with selection theory are qualitatively and quantitatively correct in the limit of small surface tension, until a time of order one, which is always in the deeply nonlinear regime.

Finally, let us recall that the presence of noise may modify the asymptotic behavior of the problem for extremely small surface tension due to the nonlinear instability of the ST finger [41]. Therefore, it is important to clearly distinguish between intrinsic dynamics and noise effects (see discussion in Sec. III B). In fact, when numerical noise is properly controlled, all numerical evidence [28,29,38,42] unambiguously shows that the ST single finger is indeed the universal attractor of the intrinsic dynamics for arbitrarily small d_0. Accordningly, only if the limit of vanishing surface tension is taken for a fixed amount of noise, then the asymptotic dynamics may appear chaotic as described in Ref. [26].

B. A dynamical solvability scenario

In Ref. [30] we pointed out for the first time the dynamical implications of the MS analysis when extended to multifinger fixed points. We pursued this extension of the steady state selection problem explicitly in Refs. [4,31], where we found that, in direct analogy to the single-finger case, the introduction of surface tension did select a discrete set of multifinger stationary states, in general with coexisting unequal fingers. Here we would like to discuss in what sense that analysis provides a dynamic solvability scenario.

Before doing that, let us briefly consider an alternative view of a possible dynamical solvability scenario (DSS) proposed by Sarkissian and Levine [25]. In Ref. [25], it was explicitly discussed with examples that exact solutions of the zero surface tension problem did behave differently from numerical integration of the small surface tension problem. At the end, the authors speculated on the possibility that surface tension could play a selective role in the sense that it could basically pick up the physically correct evolutions out of the complete set of solutions without surface tension, in direct analogy with the introduction of a small surface tension selecting a unique finger width out of the continuum of stationary solutions. Since the class of nonsingular integrable solutions is indeed vast and infinite dimensional, it is not unreasonable to expect that one could approximate any particular evolution with finite surface tension with one of those solutions for all time. However, as recently pointed out in Ref. [26], there is no simple way to determine which of those solutions is selected by any macroscopic construction. Furthermore, even if this were possible, one should still face the rather uncomfortable fact that the base of solutions defined by the superposition of logarithmic terms in the mapping, would itself correspond to unphysical (nonselected) solutions, as we have seen throughout this paper. Indeed, an initial condition defined exactly by a finite number of logarithms would have to be replaced in general by a solution with an infinite number of logarithms as the "selected" solution that the (small) finite surface tension system tracks.

From a more general point of view, a dynamical selection principle understood as "selection of trajectories" has an important shortcoming when considered within the perspective of a broader class of interfacial pattern forming systems. In fact, the solvability theory of steady state selection has turned into a general principle because its applicability to a large variety of systems, most remarkably in the context of dendritic solidification [5–7,9]. However, it is only for Laplacian growth problems that exact time-dependent solutions are known explicitly, so there would be no hope to extend the above DSS as a general principle to those other problems.

The DSS we propose here has a weaker form but it is susceptible of generalization to other interfacial pattern forming systems. The basic idea can be best expressed in words similar to those recently used by Gollub and Langer [9] to describe solvability theory in a general context. They have nicely synthesized the singular role of surface tension in the language of dynamical systems as to "whether or not there exists a stable fixed point" [9]. In this context, our DSS extends the (static) solvability scenario in the sense that the singular role of surface tension is precisely to guarantee the existence of multifinger fixed points with a saddle-point (hyperbolic) structure. We have seen that the continuum of multifinger fixed points is directly related to a nonhyperbolic structure of the equal-finger fixed points. They imply directions in phase space were the flow is marginal, and this is so to all derivative orders. While in the traditional solvability scenario the introduction of surface tension does isolate a stable fixed point (a continuum of single-finger fixed points turns into a stable one and a discrete set of unstable ones), now it isolates multifinger saddle points out of continua of multifinger solutions, as discussed in Refs. [31,4] (a continuum of n-finger fixed points turns into a hyperbolic fixed point with stable and unstable directions, and a discrete set of unstable ones). Since the saddle fixed points are defined by the degenerate n-equal-finger solutions, the stable directions of the saddle point are directly related to the stable directions of the single-finger fixed point, while the unstable directions correspond to all perturbations that break the n-periodicity of the equal-finger solution. The most important stable and unstable directions, however, are those depicted in the two-dimensional phase portraits discussed in the above section, namely the "growth" direction connecting the planar interface and the n-finger fixed point, and the "competition" direction connecting the n-finger fixed point to the single-finger fixed point.

Note that, despite the formal analogy to the single-finger solvability theory, the reference to the restoration of multifinger hyperbolicity by surface tension as *dynamical* solvability scenario is fully justified. Indeed, the local structure of the multifinger fixed point has a dramatic impact on the global (topological) structure of the phase space flow, as we have seen in simple examples. In fact, the nature of saddle points is inherently dynamical in the sense that they govern pathways in phase space as opposed to the unstable and stable fixed points that just define the origin and the end of the evolution. The existence of a small but finite surface tension thus determines a global flow structure through the selection of saddle points and it is in this sense that it "selects" the dynamics of the system.

The possibility of extension of this analysis to other interfacial pattern forming problems relies on the existence of a continuum of unequal multifinger stationary solutions with zero surface tension. The fact that in the ST case the existence of those can be associated with a simple relationship between screening due to relative tip position and relative finger width (that is, a slower areal growth rate of the screened finger is compensated by its smaller width, resulting in an equal tip velocity), one could expect that similar classes of solutions must exist in other problems, for instance, in the growth of needle crystals in the channel geometry via the connection to the fingering problem in a sector [49]. Although this point should be more carefully addressed, it seems reasonable to expect that a DSS as presented above could be generalizable, to some extent, to other physical systems.

VII. SUMMARY AND CONCLUSIONS

We have developed a dynamical systems approach to study the dynamics of the Saffman-Taylor problem, basing the analysis on the zero surface tension solutions. A minimal model has been analyzed, and from its phase flow we have concluded that it is unphysical. A detailed study of a perturbation of the minimal model within two dimensions has yielded the same conclusion. The unphysical behavior of zero surface tension solutions is a consequence of the nonhyperbolicity of the multifinger fixed points of the finite-dimensional dynamical system that they define, opposed to the saddle-point structure of the regularized problem. Accordingly, the equal-finger fixed point lacks the unstable direction that is associated with finger competition. Generalizations of the minimal model to higher dimensions and less symmetric situations confirm the generality of the conclusions reached in the two-dimensional case. We have proved that the N-logarithm class of solutions generically presents finite-time singularities if the continua of fixed points are not present. Removal of the continua of multifinger fixed points also removes the equal-finger fixed points hence finger competition as instability of the equal finger solutions is also missed. We thus conclude that an unfolding of the nonhyperbolic equal-finger fixed point does not exist within the class of integrable solutions. From the analysis of zero surface tension solutions we conclude that they are unphysical in a global sense, when sufficiently large classes of initial conditions are considered simultaneously, because they lack the correct topology of the physical flow, structured in terms of a saddle-point connection between the unstable and the stable fixed points. This does not exclude that, for some sets of initial conditions, the zero surface tension dynamics might be correct, not only qualitatively but even quantitatively, but it is not possible in practice to know it *a priori* by any simple means. We have illustrated with several examples that although the asymptotic behavior may be correct (evolution towards a single ST finger) the intermediate dynamics may be completely wrong, or even physically meaningless, such as for the existence of interface crossings. We have also illustrated the sensitivity to initial conditions when approximating physically relevant situations with different integrable solutions. We have found explicit solutions that lead to finite-time interface pinchoff in the stable configuration of the problem.

The detailed comparison of the dynamics with zero and nonzero but very small surface tension requires a careful numerical study and can be analyzed in terms of the daughter singularities formalism developed in Refs. [28,29]. As a matter of fact it can be shown that the zero surface tension problem and the vanishingly small surface tension regularization differ dramatically even in regions where the former is nonsingular, in the sense that nonzero measure regions of phase space have a different outcome of the competition (namely, which one of two competing fingers survives) in the two cases. A detailed study of this point will be presented elsewhere [38].

Finally, we propose a dynamical solvability scenario that is not only relevant in principle for viscous fingering problems but also applicable to other pattern forming problems. Within this DSS the role of surface tension as a singular perturbation is to isolate multifinger saddle points out of the continua of multifinger fixed points, as shown previously in Refs. [31,4]. This extends the traditional solvability theory applied to steady state selection, where surface tension did also isolate a unique (stable) hyperbolic fixed point out of a continuum of nonhyperbolic ones. In that case the isolated fixed point was the global attractor of the problem. In the present extension, the introduction of surface tension does isolate a unique n-equal-finger fixed point out of each continuum of n-finger fixed points, with both stable and unstable directions. By restoring this saddle-point local structure the topology of the phase space flow is modified, so the introduction of surface tension has a deep impact on the global phase-space structure of the dynamics. It is in this sense that this scenario can be considered as a *dynamical* solvability theory.

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